Does a sinc() function match the magnitude response of a tone in a Fourier Transform?

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The sinc(), aka sinx/x, is often used to describe the sidelobe behavioral model for a tone in the frequency domain. The sinc() is the FT of a rectangular function for the continuous case of the FT, and the Dirichlet kernel is the FT of the rectangular function for the discrete sampled case. As N, the number of samples in the transform, goes to infinity or the sampling interval goes to zero, the Dirichlet kernel converges to the sinc() function.

While "sinx/x" is often used as a descriptive term for the general shape of many of the common transform window functions and is often used as the general descriptor for tone sidelobe behavior, in this case we compare the specific differences between the values of the sinc() function, the tone sidelobe behavior for a tone exactly between two DFT bins, and the Dirichlet kernel.

Generate a complex-valued tone of length N:

$j := \sqrt{-1}$ Define the radical for engineering clarity.

$N := 16$ Length of the vectors.

$n := 0..N-1$ Time index for vectors.

$f_t := 5.5$ Tone frequency, in cycles/aperture, where N is the length of the aperture.

$x_n := e^{j\cdot\left(\frac{2\cdot\pi\cdot n\cdot f_t}{N}\right)}$ Define tone samples.

$k := 0..N-1$ Frequency index for transform output.

$X_k := \frac{1}{N}\cdot\sum_{n=0}^{N-1}\left(x_n\cdot e^{-j\cdot\frac{2\cdot\pi\cdot k\cdot n}{N}}\right)$ Perform a DFT.

Define sinc(x) with the exception for x = 0. This prevents a singularity at the main lobe peak.

$$\text{sinc}(x) := \text{if}\left(x = 0, 1, \left|\frac{\sin(x)}{x}\right|\right)$$

$$\text{esinc}(k, f_t) := \text{sinc}\left[\pi\cdot(k - f_t)\right]$$

Define the expected sinc() envelope considering the tone frequency.

The full definition of the Dirichlet kernel includes parameters for the sample length of the window, K, the length of the DFT, N, where N ≥ K, and the offset of the start of the window from the center (origin) of the DFT, m. For continuity with the above analysis, k is used as the frequency index.

$$\text{Dirichlet} = e^{j\left(\frac{2\cdot\pi\cdot k}{N}\right)\cdot\left[m-\frac{(K-1)}{2}\right]}\cdot\frac{\sin\left(\frac{\pi\cdot k\cdot K}{N}\right)}{\sin\left(\frac{\pi\cdot k}{N}\right)}$$

For the general, non-zero-padded transform, case the window is the length of the DFT, so that K = N and m = (K-1)/2. This makes the phase term for the exponential zero and simplifies the remaining sin() ratio.

$$\text{dkern}(k, f_t, N) := \left|\frac{\frac{1}{N}\cdot\sin\left[\pi\cdot(k - f_t)\right]}{\sin\left[\frac{\pi\cdot(k - f_t)}{N}\right]}\right|$$

Define the magnitude of the Dirichlet kernel for a symmetric, length N window. Normalizing by 1/N so that the peak is unity makes it directly comparable with the sinx/x function. The f_t term centers the response at that frequency.

$$mx_k := \left|X_k\right|$$

Compute the magnitude of the transform.

Now the magnitude of the DFT output, expected sinc(), and Dirichlet kernel profiles can be compared.

mx =

	0
0	0.07087
1	0.08085
2	0.09852
3	0.13258
4	0.21531
5	0.63764
6	0.63764
7	0.21531
8	0.13258
9	0.09852
10	0.08085
11	0.07087
12	0.06531
13	0.0628
14	0.0628
15	0.06531

$\mathrm{esinc}(k, f_t) =$

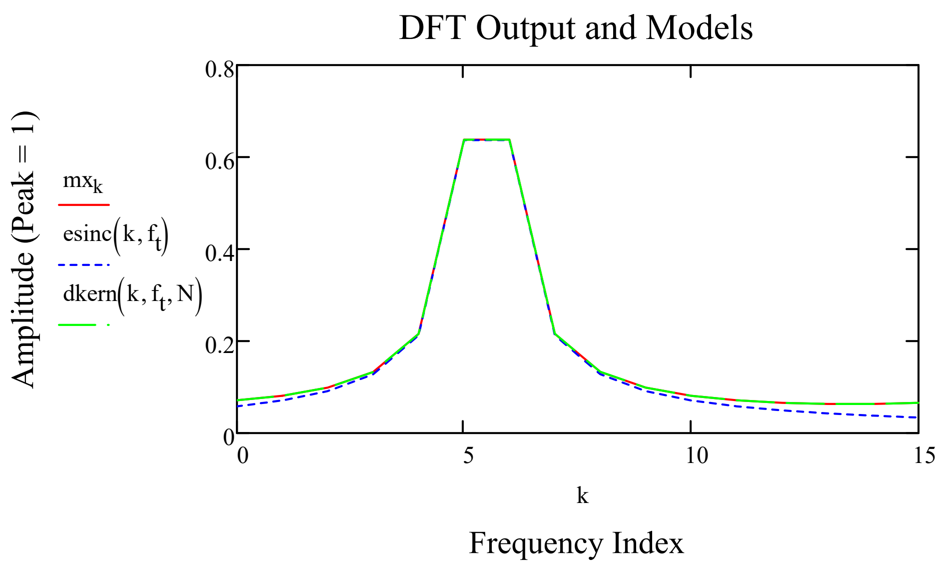
0.05787
0.07074
0.09095
0.12732
0.21221
0.63662
0.63662
0.21221
0.12732
0.09095
0.07074
0.05787
0.04897
0.04244
0.03745
0.03351

$\mathrm{dkern}(k, f_t, N) =$

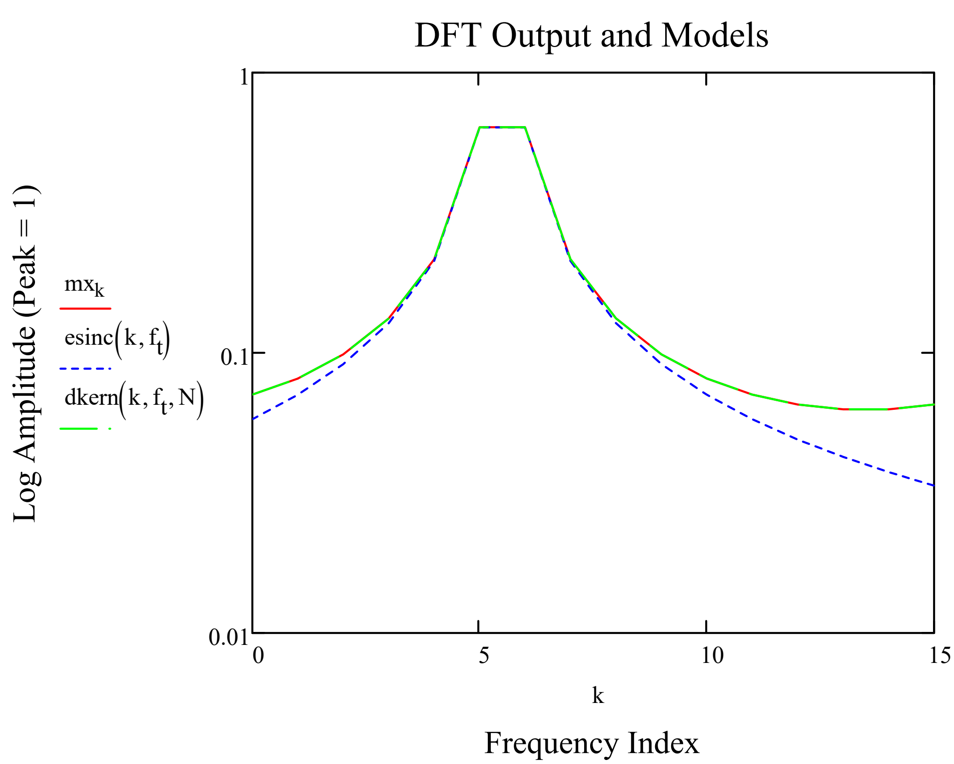
	0
0	0.07087
1	0.08085
2	0.09852
3	0.13258
4	0.21531
5	0.63764
6	0.63764
7	0.21531
8	0.13258
9	0.09852
10	0.08085
11	0.07087
12	0.06531
13	0.0628
14	0.0628
15	0.06531

The Dirichlet kernel matches the computed DFT sidelobe magnitude more closely than the sinc(), especially for small N. The symmetry of the sinx/x sidelobes is preserved in all three cases. The sidelobe behavior in all three cases are consistent in sidelobe symmetry and zero-crossing location.

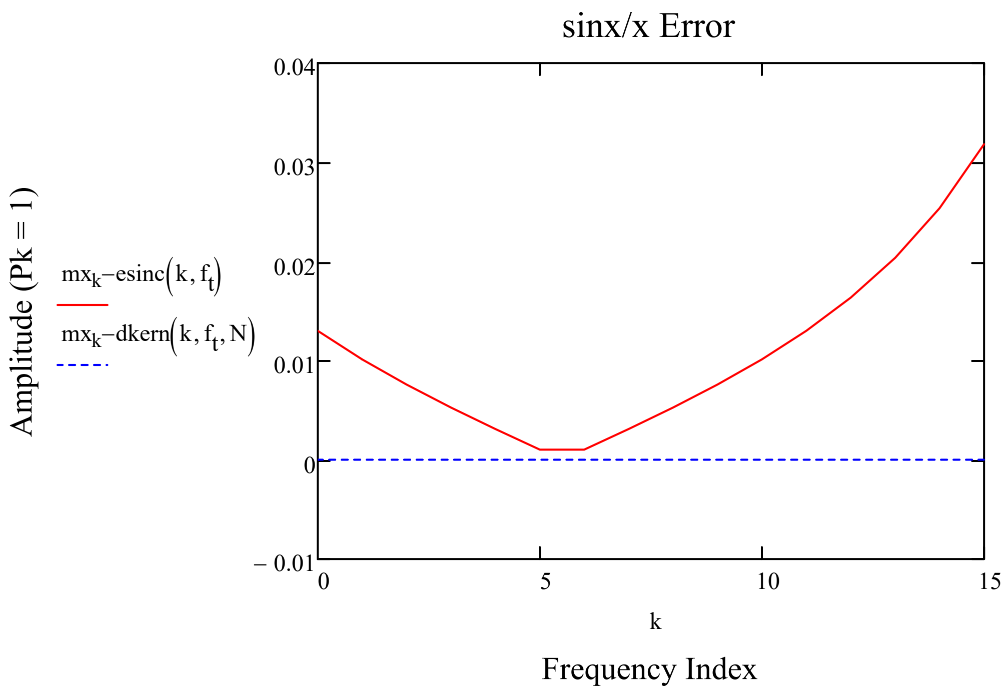
The results can be plotted to show the differences graphically. The difference between the sinx/x and the realized DFT output sidelobe levels increases with distance from the main lobe.



Rescaling the above plot with the vertical axis on a log scale helps to show that the difference in the sidelobe levels is consistent with the circular convolution of the window function with the tone, which the sinc() function is not subject to in the continuous case.



Below the error of the sinx/x to the realized sidelobe magnitude is shown as a function of frequency. The error is minimum around the main lobe and increases with distance from the main lobe. Again, this is consistent with the sidelobe profile being circular in the discrete case and linear in the continuous case.



As N increases the period of the Dirichlet kernel increases, which reduces the difference with the sinc() function in sidelobe profile.   The difference at the main lobe decreases as well.

$N := 1024$ — Length of the vectors.

$n := 0 .. N-1$ — Time index for vectors.

$f_t := 250.5$ — Tone frequency, in cycles/aperture, where N is the length of the aperture.

$x_n := e^{j\cdot\left(\frac{2\cdot\pi\cdot n\cdot f_t}{N}\right)}$ — Define tone samples.

$k := 0 .. N-1$ — Frequency index for transform output.

$X_k := \frac{1}{N}\cdot\sum_{n=0}^{N-1}\left(x_n\cdot e^{-j\cdot\frac{2\cdot\pi\cdot k\cdot n}{N}}\right)$ — Perform a DFT.

$mx_k := \left|X_k\right|$ — Compute the magnitude of the transform.

mx =

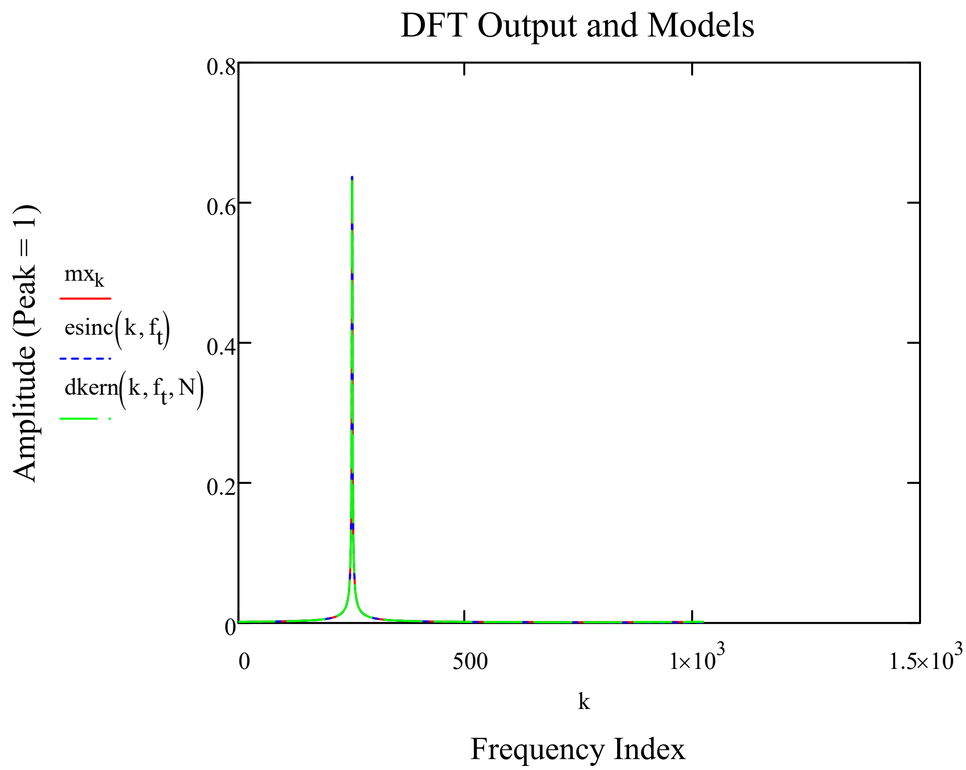
	0
0	$1.405\cdot10^{-3}$
1	$1.409\cdot10^{-3}$
2	$1.414\cdot10^{-3}$
3	$1.419\cdot10^{-3}$
4	$1.423\cdot10^{-3}$
5	$1.428\cdot10^{-3}$
6	$1.432\cdot10^{-3}$
7	$1.437\cdot10^{-3}$
8	$1.442\cdot10^{-3}$
9	$1.447\cdot10^{-3}$
10	$1.452\cdot10^{-3}$
11	$1.457\cdot10^{-3}$
12	...

$esinc(k, f_t) =$

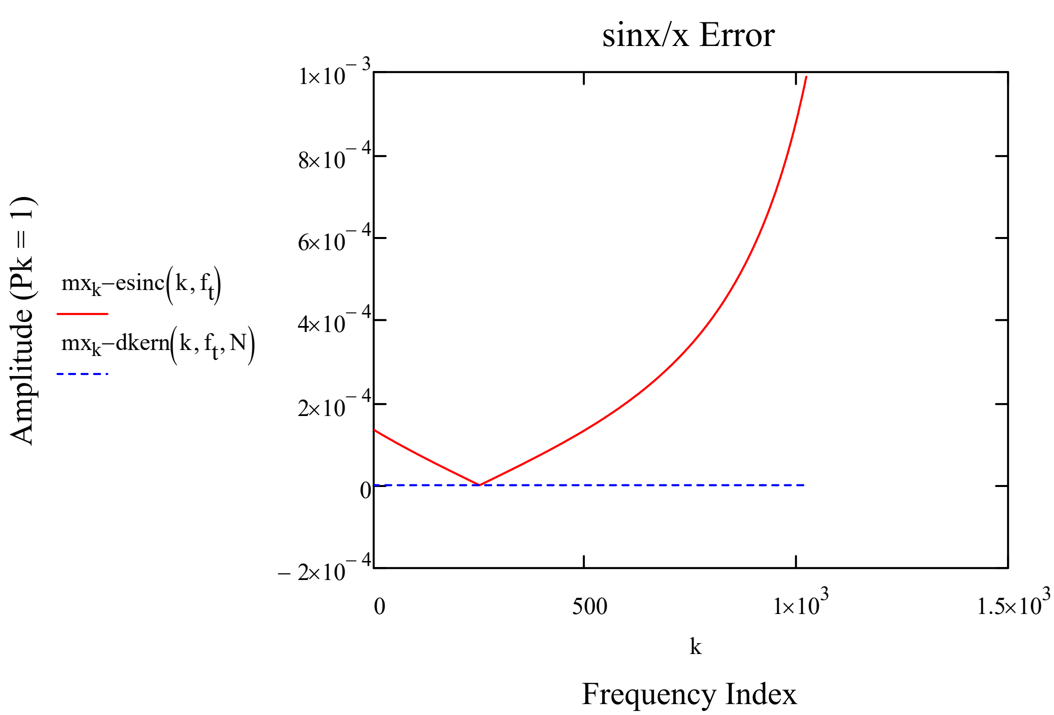
$1.271\cdot10^{-3}$
$1.276\cdot10^{-3}$
$1.281\cdot10^{-3}$
$1.286\cdot10^{-3}$
$1.291\cdot10^{-3}$
$1.297\cdot10^{-3}$
$1.302\cdot10^{-3}$
$1.307\cdot10^{-3}$
$1.313\cdot10^{-3}$
$1.318\cdot10^{-3}$
$1.324\cdot10^{-3}$
$1.329\cdot10^{-3}$
...

$dkern(k, f_t, N) =$

$1.405\cdot10^{-3}$
$1.409\cdot10^{-3}$
$1.414\cdot10^{-3}$
$1.419\cdot10^{-3}$
$1.423\cdot10^{-3}$
$1.428\cdot10^{-3}$
$1.432\cdot10^{-3}$
$1.437\cdot10^{-3}$
$1.442\cdot10^{-3}$
$1.447\cdot10^{-3}$
$1.452\cdot10^{-3}$
$1.457\cdot10^{-3}$
...

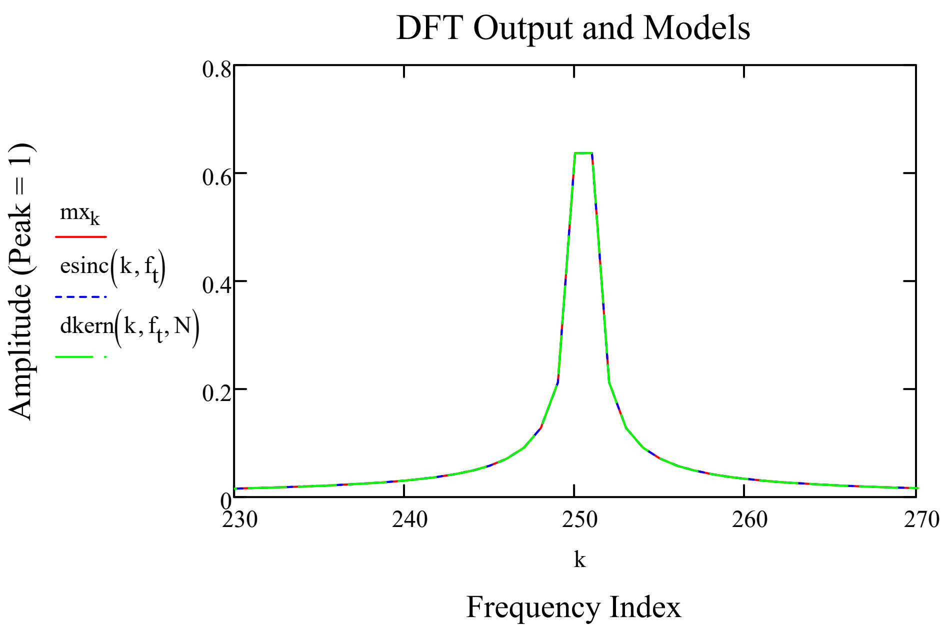


DFT Output and Models
Amplitude (Peak = 1)
0.8
0.6
0.4
0.2
0
mx_k
esinc(k, f_t)
dkern(k, f_t, N)
0
500
1×10^3
1.5×10^3
k
Frequency Index

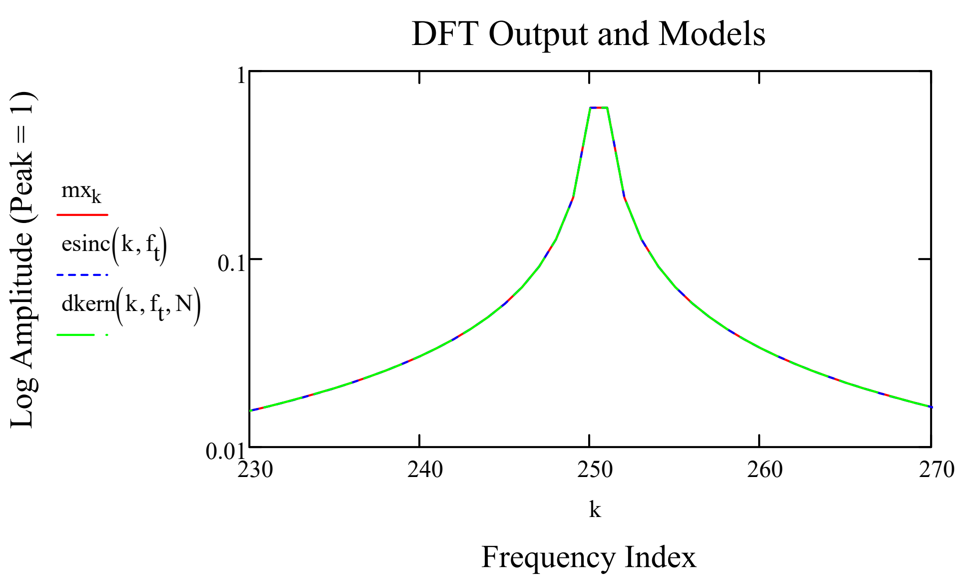


sinx/x Error
Amplitude (Pk = 1)
1×10^-3
8×10^-4
6×10^-4
4×10^-4
2×10^-4
0
-2×10^-4
mx_k-esinc(k, f_t)
mx_k-dkern(k, f_t, N)
0
500
1×10^3
1.5×10^3
k
Frequency Index

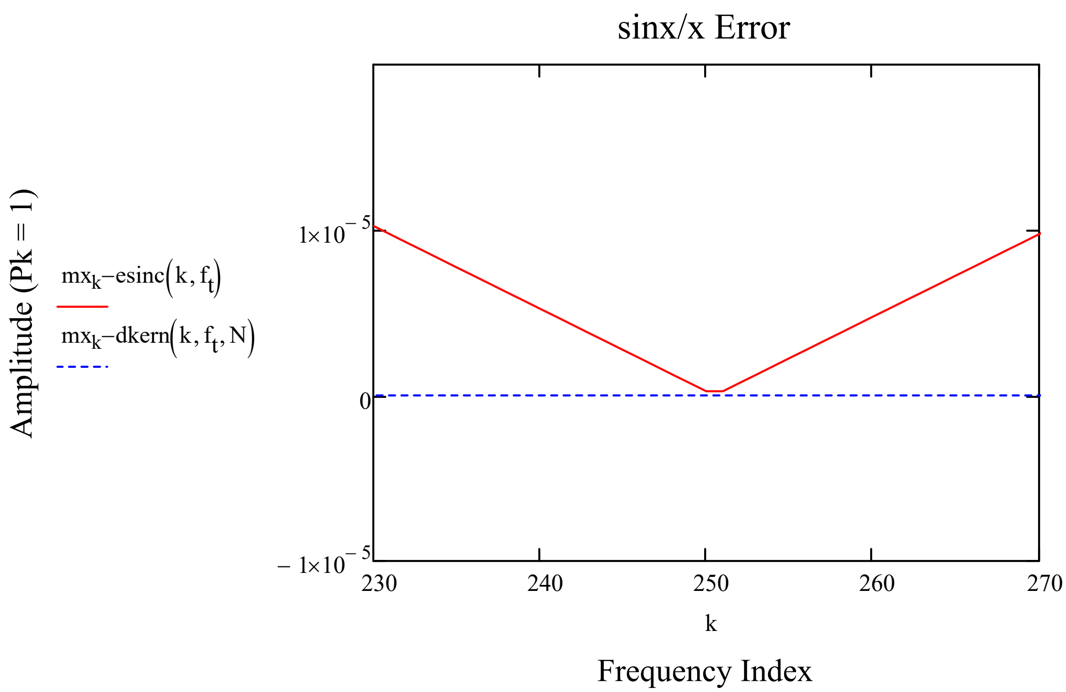
The plot below shows the large-N output zoomed-in around the peak.



Even twenty samples on either side of the main lobe the errors are small. The same plot is shown below with a log scale on the vertical axis.



The plot below shows the error between the sinc() profile and the DFT magnitude output. The error at the main lobe samples is ~2.5e-7, or -66dBc.



Analysis

Since DFT analysis of tones is usually focused on the behavior around the main lobe, the general model for DFT sidelobe behavior with a rectangular winodw is often the sinx/x or sinc() function. While this model is accurate for non-small values of N, the error in sidelobe magnitude increases with the distance from the main lobe. As N decreases or analysis interest away from the main lobe increases, the Dirichlet kernel provides a more accurate model for sidelobe magnitude.

Since the Dirichlet kernel is a sinx/x resampled in the frequency domain, the function is circular over the length of the transform. For this reason the Dirichlet kernel is sometimes called the "aliased sinc" or "aliased sinx/x" where the function can be thought of as being periodic with the length of the transform.

Reference:

R. Lyons, "Understanding Digital Signal Processing," 2nd Ed., Prentice-Hall, 2004

Note: FWIW, Rick's book is one of the few places where a proper discrete-domain treatment of the Dirichlet kernel can be found.

Appendix A

In order to show the circular, or repetitive nature of the Dirichlet kernel and the linear nature of the sinc() function, we can re-create the short-vector version of the tone (since it got nuked in the large-N example above).

$N := 16$ Length of the vectors.

$n := 0 .. N-1$ Time index for vectors.

$f_t := 5.5$ Tone frequency, in cycles/aperture, where N is the length of the aperture.

$x_n := e^{j\cdot\left(\frac{2\cdot\pi\cdot n\cdot f_t}{N}\right)}$ Define tone samples.

$k := 0 .. N-1$ Frequency index for transform output.

$X_k := \frac{1}{N}\cdot\sum_{n=0}^{N-1}\left(x_n\cdot e^{-j\cdot\frac{2\cdot\pi\cdot k\cdot n}{N}}\right)$ Perform a DFT.

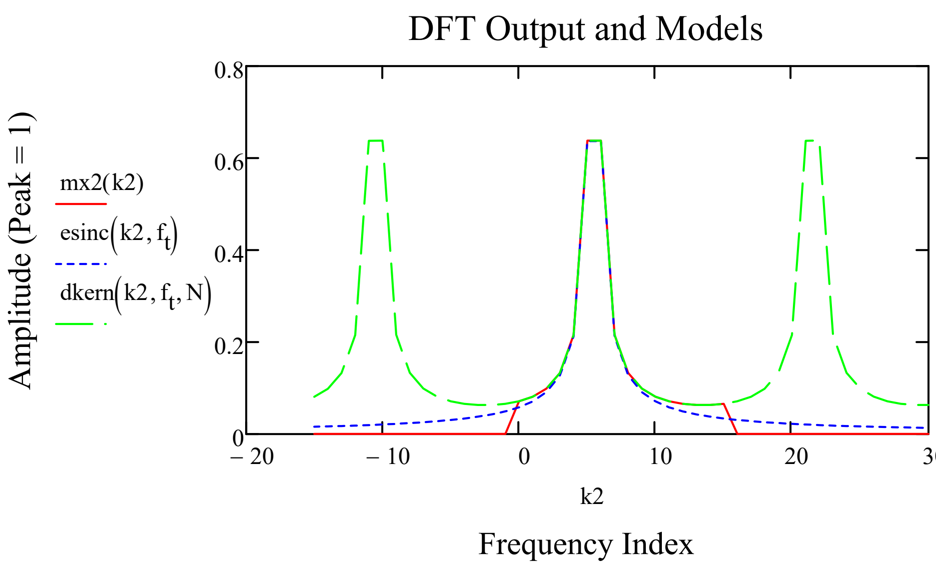
$mx_k := \left|X_k\right|$ Compute the magnitude of the transform.

Now extend the frequency range for N additional samples in either direction.

$k2 := -(N-1) .. 2\cdot(N-1)$

$mx2(k2) := \text{if}\left(k2 < 0, 0, \text{if}\left(k2 < N, mx_{k2}, 0\right)\right)$

The result shows three periods of the Dirichlet kernel, and the sinc() function extending, linearly, in either direction.



As in the case described above in the main text, the difference is somewhat more apparent with a log scale on the vertical axis.